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Proportionality in multiple dimensions to design electoral systems

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Abstract

Proportionality is a foundational principle in the election of representatives for legislative bodies. In this work, we provide an axiomatic justification for a recently proposed notion of multidimensional proportionality and extend it to incorporate two common electoral constraints: vote threshold for list eligibility and the election of the most-voted candidate in each district. Building on these results, we design methods based on multidimensional proportionality and apply them to the 2021 Chilean Constitutional Convention election, considering political lists, districts, and genders as dimensions. We evaluate the resulting apportionments according to three criteria: proportionality, representativeness, and voting power. Whereas local and global methods exhibit a natural trade-off between local and global proportionality, incorporating the election of district-level winners into methods based on 3-dimensional proportionality enables a balance between both notions while ensuring higher representativeness and a more equitable voting power. These findings are further supported by experiments on the 2023 Finnish Parliamentary Election.

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1 Introduction

The idea of proportionality is fundamental in mathematics and has been present in a wide variety of applications in many different disciplines. We usually say that two sets of values are proportional if one can be obtained by scaling the other by a common value. However, proportionality can be easily extended to the case where more than one dimension is involved, giving rise to the so-called *tensor scaling*. This has been widely studied, achieving both a theoretical understanding of it and efficient algorithms for performing it (see, e.g., Allen-Zhu et al. 2017; Deming and Stephan 1940; Rote and Zachariasen 2007). Applications of this progress are varied and include, among others, transportation planning (Evans 1970), contingency table analysis (Deming and Stephan 1940), national account systems in economies (Stone 1964), teletraffic data transport (Kruithof 1937; Krupp 1979), and stochastic matrices (Sinkhorn 1964). Tensor scaling not only naturally extends the usual notion of proportionality, but also has very strong and natural properties that one would expect of such procedure, as it has been observed by Aumann and Maschler (1985), Balinski and Demange (1989b), and Young (1987), among others.

For several applications, an additional property of the solution turns out to be relevant: integrality. Think, for example, of the case where some population is simultaneously split according to some demographic features, and a certain number of people among each group is to be selected "proportionally" to the relative presence of such group within the total population.[1] Another example, which motivates this work and will be developed in further detail, is the election of a deliberative assembly, where the candidates participate as part of a given list in a particular district and belong to a demographic group that is to be proportionally represented in the assembly. In these situations, we want to obtain integer values proportional to some input quantities. The challenge of extending proportionality to this setting gives rise to the so-called divisor methods, initially introduced in the 18th century for the elections of the recently created U.S. House of Representatives and widely used in elections until today. This notion of proportionality, based on scaling the input numbers by a common value and rounding the result, was extended to the case of two dimensions by Balinski and Demange (1989b). In recent work, Cembrano et al. (2022) showed that it is possible to further extend the same approach to the case of arbitrary dimensions, as long as we allow some constant deviations from the total number assigned to each group. These methods use the same approach as divisor methods and constitute a natural way of adapting tensor scaling to the case where integer values are required.

In this work, we focus on the applications of multidimensional proportionality to the election of representatives, in the so-called apportionment problem. The main motivation is the growing need to incorporate dimensions beyond the classic political and geographical aspects in elections, as modern societies become more complex. The increasing need for such methods has driven increasing attention to the subject in the fields of social choice, mathematics, and computer science (Flanigan et al. 2021; Lang and Skowron 2018; Pukelsheim 2017). In practice, New Zealand's parliament has included ethnic representation for more than 50 years, while Bosnia and Herzegovina's

[1] For instance, this could be done with the purpose of performing some poll or census.

Parliament was proposed to include a division of three types of “Constituent People”: Bosniacs, Croats, and others (Demange 2013). A recent case, which we introduce in more detail below and use as a running example throughout the paper, is provided by the election of the Chilean Constitutional Convention in 2021.

After civil unrest broke out in Chile on October 18, 2019, a political agreement to write a new political constitution was achieved. The agreement was reaffirmed by a referendum approved with 78% of the votes on October 25, 2020, and the voting process to elect the members of the upcoming Constitutional Convention took place on May 15–16, 2021. This election drew particular attention due to two constraints on the representative body to be elected. First, 17 of the 155 seats of the convention were reserved for 10 different ethnic groups.[2] Second, a remarkable feature of the election was the incorporation of gender balance in the convention as a way to ensure the proportional representation of women, historically underrepresented in Chilean politics: In the 2021 Parliament election, they achieved their maximum historical representation by taking 34% of the seats in the lower house. This is a reflection of a global situation: The average percentage of women in parliaments around the world has increased from 11.3% to 26.9% between 1995 and 2023, according to the Inter-Parliamentary Union (2024), which is still far from a balanced representation. In response to this, various nations like Germany, Belgium, Spain, and, more recently, Chile, have tried to promote gender parity in the legislative power through different correction mechanisms, mostly regarding party-level candidate quotas, both in proportional and majoritarian electoral methods.

Our contribution. In this paper, we study and develop methods to handle proportionality in multiple dimensions when electing a house of representatives. These methods build upon the notion of multidimensional proportionality introduced by Cembrano et al. (2022), which naturally extends that for one and two dimensions (Balinski and Young 2010; Balinski and Demange 1989b) and is based on scaling each element of each dimension by a multiplier and rounding the result. We begin by showing two results that support the use of this notion of multidimensional proportionality in real-life elections. On the one hand, we prove that it is the rounded version of the unique notion of multidimensional proportionality that satisfies appropriate versions of three widely-studied axioms in the social choice and apportionment literature: exactness, consistency, and homogeneity. This result extends that of Balinski and Demange (1989b) for the 2-dimensional case. Unlike the one-dimensional case, in two or more dimensions the correct notion of fractional proportionality to begin with is not trivially defined, hence the importance of these axiomatic results. On the other hand, we show that it is possible to extend this notion of multidimensional proportionality to the case where there are lower or upper bounds on the number of seats a specific tuple of elements should receive. These bounds may come, for instance, from the fact that a party must receive at least one seat in a certain district due to having the top-voted candidate, or no seats due to not reaching a certain threshold of votes. Similarly to Cembrano et al. (2022), we show that the existence of proportional apportionments deviating up to a constant α_i from the prespecified number of seats

[2] In Chile, around two million people identify themselves as part of these groups, according to the 2017 census.

in each dimension $i \in \{1, \ldots, d\}$ is guaranteed, provided the feasibility of a linear program and the inequality $\sum_{i=1}^{d} 1/(\alpha_i + 2) \leq 1$.

We then exploit the aforementioned results to propose and study several electoral methods based on 3-dimensional proportionality (abbreviated as 3-proportionality in what follows). They are defined in the context of the election of the Chilean Constitutional Convention so that the dimensions correspond to political lists, where each list should receive a number of seats proportional to its votes, geographic distribution, where each district should receive a number of seats prespecified by law, and gender, where each gender should receive half of the seats. However, we remark that our methods can be directly applied in other contexts, which we showcase by applying them in the context of the 2023 Parliamentary Election in Finland. Besides the method obtained by directly applying the notion of multidimensional proportionality of Cembrano et al. (2022), we introduce two additional features in light of our theoretical results: a threshold over the votes obtained by a list in order to be eligible and the election of the top-voted candidate of each district (also known as *plurality* constraints). These features aim to limit the segmentation of political parties and to incorporate a majoritarian component in proportional representation methods, respectively, and are often used together in countries such as Germany, New Zealand, the Republic of Korea, and Mexico. These countries used thresholds of either 3% or 5%; we employ the former in our simulations. As relevant findings, we observe that deviations from the prescribed number of seats in 3-proportional methods are significantly smaller than the worst-case theoretical guarantees. In addition, 3-proportional methods pose a fundamental trade-off between local and global proportionality and induce a political distribution closer to the well-known principle of *one person, one vote* compared to methods that operate locally. Overall, the election of top-voted candidates combined with a threshold over the votes obtained by a list appears as a reasonable midpoint while ensuring high levels of representativeness, in terms of the average number of votes of the elected candidates.

Further related work Tensor scaling, in particular the two-dimensional version, has been widely studied since the seminal work by Sinkhorn (1964); some relevant works are those by Sinkhorn and Knopp (1967), Rothblum and Schneider (1989), and Nemirovski and Rothblum (1999). Recent works improving the analysis of known algorithms and developing faster ones include those by Allen-Zhu et al. (2017); Chakrabarty and Khanna (2021); Cohen et al. (2017). Particularly relevant for our framework, where integrality is required, are the axiomatic and algorithmic results by Balinski and Demange (1989a, b); their algorithm was then analyzed by Kalantari et al. (2008). For an overview of this area, we refer the reader to the survey by Idel (2016).

The apportionment problem, and in particular divisor methods, have been extensively studied from disciplines such as operations research, computer science, and political science; the books by Balinski and Young (2010) and Pukelsheim (2017) provide historical and theoretical surveys of this topic. In their seminal work, Balinski and Demange (1989a, b) extended the notion of proportionality and divisor methods to the case in which the apportionment is ruled by two dimensions, e.g., political parties and geographic districts, studying this extension from an axiomatic and algorithmic point of view. Balinski (2008), as well as Balinski and González (1997), proposed variants of this method, while Maier et al. (2010) conducted a real-life benchmark study of biproportional apportionment and its variants, including the election of top-voted candidates in each district. Rote and Zachariasen (2007) and Gaffke and Pukelsheim (2008a, b) studied biproportional apportionment via network flow formulations, a tool studied in more general electoral settings by Pukelsheim et al. (2012). Recently, Cembrano et al. (2022) extended the biproportional approach to an arbitrary number of dimensions using tools from discrepancy theory, showing that, although proportional apportionments do not always exist, they do if small deviations are allowed. Other recent works regarding the use of tools from optimization and rounding in the development of (randomized) apportionment methods include those by Gölz et al. (2022) and Correa et al. (2024). New perspectives on social choice and apportionment methods have also been studied by Balinski and Laraki (2007, 2011) and Serafini (2020).

2 Preliminaries

In the one-dimensional apportionment problem, we are given a vector $\mathcal{V} \in \mathbb{N}^n$ containing the votes obtained by each party $i \in \{1, \dots, n\}$ and a house size $H \in \mathbb{N}$. The goal is to find an allocation of seats to parties that is proportional to the votes. We let $f^1(\mathcal{V}, H) = \lambda\mathcal{V}$ denote the 1-*dimensional fair share* for input $(\mathcal{V}, H)$, where $\lambda = H / \sum_{i=1}^n \mathcal{V}_i$. Based on divisor methods—which constitute our notion of integral proportionality in what follows—we let $\mathcal{A}^1(\mathcal{V}, H)$ be the set of 1-*dimensional proportional apportionments*, which are the integral vectors obtained by scaling $\mathcal{V}$ by a multiplier and rounding, i.e., the set of vectors $x \in \mathbb{N}^n$ satisfying

$$x_i \in [\![\lambda\mathcal{V}_i]\!] \qquad \text{for every } i \in \{1, \dots, n\} \text{ and} \tag{1}$$

$$\sum_{i=1}^n x_i = H \tag{2}$$

for some $\lambda > 0$, where $[\![\cdot]\!]$ represents a rounding rule. Throughout this paper, we focus for the sake of simplicity on the Jefferson/D'Hondt method and its generalizations to multiple dimensions, so that $[\![\cdot]\!]$ will denote the downwards rounding rule.[3] More precisely, we define $[\![0]\!] = \{0\}$, $[\![t]\!] = \{r\}$ when $t \in (r, r+1)$ and $[\![t]\!] = \{r-1, r\}$ when $t = r > 0$.[4] When vectors $I, U \in \mathbb{N}^n$ such that $\sum_{i=1}^n I_i \leq H \leq \sum_{i=1}^n U_i$ are given as well, we let $\mathcal{A}^1(\mathcal{V}, H, I, U)$ be the set of vectors $x \in \mathbb{N}^n$ satisfying (2) and

$$x_i \in \operatorname{mid}\left([\![\lambda\mathcal{V}_i]\!] \cup \{I_i, U_i\}\right) \qquad \text{for every } i \in \{1, \dots, n\} \tag{3}$$

for some $\lambda > 0$, where $\operatorname{mid}(S)$ represents the median value of S, for $S \subset \mathbb{R}$. Note that, in a slight abuse of notation, fixing $I_i = 0$ and $U_i = H$ for each i we have $\mathcal{A}^1(\mathcal{V}, H) = \mathcal{A}^1(\mathcal{V}, H, I, U)$ for every $\mathcal{V} \in \mathbb{N}^n$ and $H \in \mathbb{N}$. The described apportionments are

3 However, our theoretical results are valid for any other rounding rule.

4 This set-valued rounding is needed to ensure the existence of proportional apportionments. Note that this may lead to ties between several apportionment vectors, which will also be the case for our multidimensional methods.

guaranteed to exist and can be found through fast combinatorial algorithms or linear programming (Balinski and Young 2010; Reitzig and Wild 2024).

The basic definition of one-dimensional apportionments—without the presence of component-wise bounds—was adapted to two dimensions by Balinski and Demange (1989a, b) and to three and more dimensions by Cembrano et al. (2022); we here introduce the solution concept in full generality and discuss its existence and restrictions later on. In the d-dimensional apportionment problem, we are given sets $N_1, \dots, N_d$ and an integer matrix $\mathcal{V} \in \mathbb{N}^{\prod_{i=1}^{d} N_i}$, with one natural value for each tuple $e \in N_1 \times \cdots \times N_d$ representing the votes obtained by the candidates of this tuple. We define $E(\mathcal{V}) = \left\{e \in \prod_{i=1}^{d} N_i : \mathcal{V}_e > 0\right\}$.[5] We are also given values $m_v \in \mathbb{N}$ for every $v \in N_i$ and every $i \in \{1, \dots, d\}$ determining the number of seats to be assigned to each element, values $I_e, U_e \in \mathbb{N}$ for every $e \in E(\mathcal{V})$ restricting the number of seats to be assigned to each tuple, and a house size $H \in \mathbb{N}$, satisfying

$$\sum_{e \in E(\mathcal{V}): e_i = v} I_e \leq m_v \leq \sum_{e \in E(\mathcal{V}): e_i = v} U_e \qquad \text{for every } i \in \{1, \dots, d\} \text{ and } v \in N_i, \text{ and} \tag{4}$$

$$\sum_{v \in N_i} m_v = H \qquad \text{for every } i \in \{1, \dots, d\}. \tag{5}$$

As the natural extension of the notion of fractional proportionality to the case of multiple dimensions, we say that a tensor x is a *d-dimensional fair share* if, for every $i \in \{1, \dots, d\}$ and $v \in N_i$, there exists a strictly positive value λ_v such that the following conditions hold:

$$\sum_{e \in E(\mathcal{V}): e_i = v} x_e = m_v \qquad \text{for every } i \in \{1, \dots, d\} \text{ and } v \in N_i, \text{ and} \tag{6}$$

$$x_e = \mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i} \qquad \text{for every } e \in E(\mathcal{V}). \tag{7}$$

The existence and uniqueness of such a tensor are guaranteed by standard convex programming techniques; see the discussion at the end of Sect. 3.1 for the details. We thus denote this unique tensor by $f^d(\mathcal{V}, m, H)$. As in the one-dimensional case, but allowing small deviations $\alpha \in \mathbb{N}^d$ from the prespecified sums, we say that an integer tensor x is an *α-approximate d-proportional apportionment* if, for every $i \in \{1, \dots, d\}$ and $v \in N_i$, there exists a strictly positive value λ_v such that the following conditions hold:

$$m_v - \alpha_i \leq \sum_{e \in E(\mathcal{V}): e_i = v} x_e \leq m_v + \alpha_i \qquad \text{for every } i \in \{1, \dots, d\} \text{ and } v \in N_i, \tag{8}$$

[5] This definition will make the notation easier given the assumption that any tuple e with $\mathcal{V}_e = 0$ must not receive any seat, consistently with any definition of a rounding rule.

$$x_e \in \operatorname{mid}\left(\left[\!\left[\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}\right]\!\right] \cup \{I_e, U_e\}\right) \qquad \text{for every } e \in E(\mathcal{V}). \tag{9}$$

We denote the set of tensors x satisfying (8)-(9) by $\mathcal{A}^d(\mathcal{V}, m, H, I, U; \alpha)$. If a tensor x satisfies this definition with $\alpha_i = 0$ for every $i \in \{1, \ldots, d\}$, we just say that x is a d-proportional apportionment for this instance, and denote the set of such tensors simply as $\mathcal{A}^d(\mathcal{V}, m, H, I, U)$. Further, when $I_e = 0$ and $U_e = H$ for every $e \in E(\mathcal{V})$, we omit these arguments as before. Cembrano et al. (2022, Theorem 4) proved that, under mild conditions,[6] an α-approximate d-proportional apportionment is guaranteed to exist whenever $I_e = 0$ and $U_e = H$ for every $e \in E(\mathcal{V})$ and $\sum_{i=1}^{d} 1/(\alpha_i + 2) \leq 1$, and it can be found by using linear programming techniques. For instance, when $d = 3$ the above condition is satisfied when $\alpha_1 = \alpha_2 = \alpha_3 = 1$, or when $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = 4$. In Sect. 3, we extend this existence result to the case of arbitrary vectors I and U.

We often need to aggregate the components or restrict to a certain entry of a 3-dimensional tensor in our methods; we write a subindex "+" for the former case and the subindex with the entry for the latter. We write a subindex $\cdot$ when a certain dimension is preserved. For example, for a tensor $\mathcal{V} \in D \times L \times G$ and a fixed $d \in D$, $\mathcal{V}_{d,\cdot,+}$ represents a vector in $\mathbb{N}^{|L|}$ whose ℓ-th component is $\sum_{g \in G} \mathcal{V}_{d\ell g}$.

3 Theoretical results

In this section, we present our theoretical contribution. It consists of two extensions of previous results that we consider relevant for a better understanding and application of apportionment methods based on multidimensional proportionality. The first one, introduced in Sect. 3.1, is a characterization of the multidimensional fair share tensor as the only one satisfying a set of three natural axioms, which generalizes the result by Balinski and Demange (1989b) for $d = 2$ and supports our notion of multidimensional proportionality. The second one, presented in Sect. 3.2, states the existence of approximate d-dimensional apportionments with arbitrary component-wise bounds, thus extending the result by Cembrano et al. (2022) and providing the basis for the application of our methods defined in Sects. 4.4 and 4.5.

3.1 Multidimensional proportional allocations: an axiomatic approach

In this section, we provide an axiomatic characterization of the d-dimensional fair share for arbitrary d, thereby justifying the notion of integral proportionality in multiple dimensions we stick to, which can be understood as a rounding of fractional proportional tensors.

Given sets $N_1, \ldots, N_d$, values m_v for every $i \in \{1, \ldots, d\}$ and $v \in N_i$, and $H \in \mathbb{N}$, we define the region of feasible allocations as the set

[6] These conditions are given by the feasibility of a linear program similar to the one we introduce in Sect. 3.

$$R(m,H)=\left\{f\in\mathbb{R}^{\prod_{i=1}^{d}N_i}:\sum_{e\in\prod_{i=1}^{d}N_i:e_i=v}f_e=m_v\ \text{ for all } i\in\{1,\dots,d\} \text{ and } v\in N_i\right\}.$$

An *allocation method* F is a mapping from an instance given by an integer vote matrix $\mathcal{V}$ with entries in $\prod_{i=1}^{d}N_i$, values m_v for every $i\in\{1,\dots,d\}$ and $v\in N_i$, and $H\in\mathbb{N}$, to a nonempty subset $F(\mathcal{V},m,H)$ of $R(m,H)$. We need some additional notation before introducing the axioms. Consider $f\in F(\mathcal{V},m,H)$ and subsets $N_1'\subseteq N_1,\dots,N_d'\subseteq N_d$. Let $E'=\Pi_{i=1}^{d}N_i'$, let $y_{E'}$ be the subtensor of y restricted to E', and let $m'(f)$ be such that $(m'(f))_v$ is defined as follows for each $i\in\{1,\dots,d\}$ and $v\in N_i'$:

$$(m'(f))_v=\sum_{e\in E':e_i=v}f_e=m_v-\sum_{e\notin E':e_i=v}f_e.$$

We now formally introduce three axioms for allocation methods.

(I) **(Exactness)** If there exists $\delta\in\mathbb{R}$ such that $\delta\mathcal{V}\in R(m,H)$, then $F(\mathcal{V},m,H)=\{\delta\mathcal{V}\}$.

(II) **(Consistency)** If $f\in F(\mathcal{V},m,H)$, the following holds:

(a) For every family of sets $N_1'\subseteq N_1,\dots,N_d'\subseteq N_d$, we have that

$$f_{E'}\in F\left(\mathcal{V}_{E'},m'(f),\sum_{e\in E'}f_e\right).$$

(b) If $g\in F(\mathcal{V}_{E'},m'(f),\sum_{e\in E'}f_e)$, then the matrix defined to be equal to g on E' and f elsewhere belongs to $F(\mathcal{V},m,H)$.

(III) **(Homogeneity)** Suppose that $i\in\{1,\dots,d\}$ and $v_1,v_2\in N_i$ are such that $\mathcal{V}_e=\mathcal{V}_{e'}$ for every pair $e,e'\in E$ with $e_i=v_1$, $e_i'=v_2$, and $e_j=e_j'$ for every $j\neq i$. Furthermore, suppose that $m_{v_1}=m_{v_2}$. Then, for every $f\in F(\mathcal{V},m,H)$ it holds $f_e=f_{e'}$ for every $e,e'\in E$ with $e_i=v_1$, $e_i'=v_2$, and $e_j=e_j'$ for every $j\neq i$.

We denote by F^* the allocation method that maps every instance $(\mathcal{V},m,H)$ to a singleton containing its fair share tensor $f^d(\mathcal{V},m,H)$. The following theorem provides an axiomatic characterization of this method, extending that by Balinski and Demange (1989b) for the case of $d=2$.

Theorem 1 *F^* is the unique allocation method satisfying exactness, consistency, and homogeneity.*

The following lemma is key for proving uniqueness. It states that every allocation method satisfying consistency and homogeneity outputs a single allocation for every instance, and its output is invariant under scaling of the vote tensor in a single dimension. To prove it, we take any instance and consider a modified vote tensor consisting of the original vote tensor and a scaled version of it according to a single dimension, next to each other. By applying consistency and homogeneity, we can conclude that the solutions for each of them must be the same (and unique).

Lemma 1 *Let F be an allocation method that satisfies consistency and homogeneity, and let $(\mathcal{V}, m, H)$ be an instance such that $\mathcal{V}$ has entries in $\prod_{j=1}^{d} N_j$. Let $i \in \{1, \ldots, d\}$ and $\mu_v > 0$ for each $v \in N_i$ be arbitrary, and let $\mathcal{V}' \in \prod_{j=1}^{d} N_j$ be defined as*

$$\mathcal{V}'_e = \mu_{e_i} \mathcal{V}_e \quad \text{for every } \; e \in \prod_{j=1}^{d} N_j.$$

Then, $|F(\mathcal{V}, m, H)| = 1$ and $F(\mathcal{V}, m, H) = F(\mathcal{V}', m, H)$.

Proof Let $\mathcal{V},\ i,\ \mu_v$ for each $v \in N_i$, and $\mathcal{V}'$ be defined as in the statement. Let N'_i be a copy of N_i, so that we denote by $v' \in N'_i$ the copy of $v \in N_i$. Consider the related problem $(\mathcal{V}'', m'', 2H)$, where $\mathcal{V}'' \in \prod_{j=1}^{i-1} N_j \times (N_i \cup N'_i) \times \prod_{j=i+1}^{d} N_j$ is defined as

$$\mathcal{V}''_e = \begin{cases} \mathcal{V}_e & \text{if } e_i \in N_i, \\ \mu_{e_i} \mathcal{V}_e & \text{if } e_i \in N'_i, \end{cases}$$

for every e, $m''_v = m_v$ for every $j \in \{1, \ldots, d\}$ and $v \in N_j$, and $m''_{v'} = m_v$ for every $v \in N_i$. Let $E = \prod_{j=1}^{d} N_j$ denote the tuples whose ith entry is in N_i and $E' = \prod_{j=1}^{i-1} N_j \times N'_i \times \prod_{j=i+1}^{d} N_j$ denote the tuples whose ith entry is in N'_i. Let finally $f \in F(\mathcal{V}'', m'', 2H)$.

By homogeneity, we have that $f_e = f_{e'}$ for every e, e' such that $e_i = v \in N_i,\ e'_i = v' \in N'_i$, and $e_j = e'_j$ for every $j \neq i$. By consistency, the subtensors of f restricted to these sets are valid allocations for the restricted inputs, i.e., $f_E \in F(\mathcal{V}, m, H)$ and $f_{E'} \in F(\mathcal{V}', m, H)$. Moreover, if there was $g_E \in F(\mathcal{V}, m, H)$ with $g_E \neq f_E$, then the tensor defined to be equal to g_E on E and to $f_{E'}$ on E' would also be in $F(\mathcal{V}'', m'', 2H)$, contradicting homogeneity. The existence of $g_{E'} \in F(\mathcal{V}', m, H)$ with $g_{E'} \neq f_{E'}$ yields an analogous contradiction. We obtain the two claims stated in the theorem from these facts: $|F(\mathcal{V}, m, H)| = 1$ and $F(\mathcal{V}, m, H) = F(\mathcal{V}', m, H)$. □

We are now ready to prove Theorem 1. For one direction, we explicitly check the axioms for the fair share method. For the other direction, we apply the previous lemma, once for each dimension, to move from an arbitrary instance to a scaled instance where the tensor corresponds precisely to the fair share vector and, due to the lemma, the output has not changed. Exactness then implies that the fair share must be the output allocation.

Proof of Theorem 1 We first prove that the fair share method F^* satisfies axioms (I)-(III), and then we prove that it is the unique one satisfying them. To prove the former, consider an arbitrary instance $(\mathcal{V}, m, H)$ and let $f^* = f^d(\mathcal{V}, m, H)$ denote the fair share tensor for this instance, for ease of notation. Let λ_v for each $i \in \{1, \ldots, d\}$ and $v \in N_i$ be multipliers that define f^*.

Exactness. Assume that $\delta \in \mathbb{R}$ is such that $\delta\mathcal{V} \in R(m, H)$. By taking $\lambda_v = \delta^{1/d}$ for every $i \in \{1, \ldots, d\}$ and $v \in N_i$, we have $\mathcal{V}_e \prod_{i=1}^d \lambda_{e_i} = \delta\mathcal{V}_e$ for each $e \in \prod_{i=1}^d N_i$. Therefore, we conclude from the uniqueness of the fair share tensor that $f^* = \delta\mathcal{V}$ and thus $F^*(\mathcal{V}, m, H) = \{\delta\mathcal{V}\}$.
Consistency. Let $N_1' \subseteq N_1, \ldots, N_d' \subseteq N_d$ be arbitrary sets and $E' = \prod_{i=1}^d N_i'$, and observe that for each $i \in \{1, \ldots, d\}$ and $v \in N_i'$ we have that

$$\sum_{\substack{e \in \prod_{j=1}^d N_j': \\ e_i = v}} f_e^* = \sum_{\substack{e \in \prod_{j=1}^d N_j: \\ e_i = v}} f_e^* - \sum_{\substack{e \notin \prod_{j=1}^d N_j': \\ e_i = v}} f_e^* = (m'(f^*))_v.$$

Since $f_e^* = \mathcal{V}_e \prod_{i=1}^d \lambda_{e_i}$ for every $e \in E'$, we conclude that

$$f_{E'}^* \in F^*\left(\mathcal{V}_{E'}, m', \sum_{e \in E'} f_e^*\right),$$

which proves part (a). For part (b), since fair share tensors are unique, the fact that $g \in F^*(\mathcal{V}_{E'}, m', \sum_{e \in E'} f_e^*)$ implies $g = f_{E'}^*$, so it is straightforward that the matrix defined to be equal to g on E' and f^* elsewhere belongs to $F^*(\mathcal{V}, m, H)$.

Homogeneity. Let $i \in \{1, \ldots, d\}$ and $v_1, v_2 \in N_i$ be such that $\mathcal{V}_e = \mathcal{V}_{e'}$ for every pair $e, e' \in E$ with $e_i = v_1,\ e_i' = v_2$, and $e_j = e_j'$ for every $j \neq i$. Furthermore, suppose that $m_{v_1} = m_{v_2}$. Suppose we had $f_e^* \neq f_{e'}^*$ for some $e, e' \in E$ with $e_i = v_1,\ e_i' = v_2$ and $e_j = e_j'$ for every $j \neq i$. This would imply $\lambda_{v_1} \neq \lambda_{v_2}$; suppose w.l.o.g. $\lambda_{v_1} > \lambda_{v_2}$. This yields

$$\begin{aligned}
\sum_{\substack{e \in \prod_{j=1}^d N_j: \\ e_i = v_1}} f_e^* &= \sum_{\substack{e \in \prod_{j=1}^d N_j: \\ e_i = v_1}} \left(\mathcal{V}_e \prod_{j=1}^d \lambda_{e_j}\right) = \sum_{\substack{e \in \prod_{j=1}^d N_j: \\ e_i = v_1}} \left(\mathcal{V}_e \lambda_{v_1} \prod_{j \in \{1,\ldots,d\} \setminus \{i\}} \lambda_{e_j}\right) \\
&> \sum_{\substack{e \in \prod_{j=1}^d N_j: \\ e_i = v_2}} \left(\mathcal{V}_e \lambda_{v_2} \prod_{j \in \{1,\ldots,d\} \setminus \{i\}} \lambda_{e_j}\right) \\
&= \sum_{\substack{e \in \prod_{i=1}^d N_j: \\ e_i = v_2}} \left(\mathcal{V}_e \prod_{j=1}^d \lambda_{e_j}\right) = \sum_{\substack{e \in \prod_{i=1}^d N_j: \\ e_i = v_2}} f_e^*,
\end{aligned}$$

where the inequality holds both from $\lambda_{v_1} > \lambda_{v_2}$ and from the fact that $\mathcal{V}_e = \mathcal{V}_{e'}$ for every pair $e, e' \in E$ with $e_i = v_1,\ e_i' = v_2$, and $e_j = e_j'$ for every $j \neq i$. This contradicts (6) in the definition of a fair share tensor since $m_{v_1} = m_{v_2}$, so we conclude that $f_e^* = f_{e'}^*$ for every $e, e' \in E$ with $e_i = v_1,\ e_i' = v_2$, and $e_j = e_j'$ for every $j \neq i$.

Now, let F be an allocation method that satisfies axioms (I)-(III) and let $(\mathcal{V}, m, H)$ be an arbitrary instance. For every $i \in \{1, \ldots, d\}$ and every $v \in N_i$, let λ_v be fair share multipliers, i.e., values such that f^* defined as $f^*_e = \mathcal{V}_e \prod_{i=1}^d \lambda_{e_i}$ for every $e \in \prod_{i=1}^d N_i$ satisfies (6)-(7). Define $\mathcal{V}'$ as $\mathcal{V}'_e = \mathcal{V}_e \prod_{i=1}^d \lambda_{e_i}$ for every $e \in \prod_{i=1}^d N_i$. Applying the result from Lemma 1 d times with $\mu_v = \lambda_v$ for every $i \in \{1, \ldots, d\}$ and $v \in N_i$, we obtain that $F(\mathcal{V}', m, H) = F(\mathcal{V}, m, H)$. But $\mathcal{V}' \in R(m, H)$, thus from exactness we have that $F(\mathcal{V}', m, H) = \{\mathcal{V}'\}$. We conclude that $F(\mathcal{V}, m, H) = \{\mathcal{V}'\} = \{f^*\}$. □

We remark that the fair share tensor can be found by solving a convex optimization program, whose constraints correspond to f being an element of the polytope $R(m, H)$ and whose convex objective function is

$$\sum_{e \in \prod_{i=1}^d N_i} \left(\log \left(\frac{f_e}{\mathcal{V}_e} \right) - 1 \right).$$

By standard KKT optimality conditions, it can be verified that the optimal solution of such a program is the d-dimensional fair share (Idel 2016).

3.2 Approximate 3-proportional apportionments with plurality

In this section, we extend the existential and algorithmic results from Cembrano et al. (2022) for the case of apportionments with lower and upper bounds on each entry. This result will guarantee the correctness of our methods with plurality constraints and may be of independent interest for other methods incorporating bounds on the number of seats that certain entries should be allocated.

In order to formalize this extension, we consider the following integer linear program inspired by the network flow approach introduced by Rote and Zachariasen (2007) for matrix scaling, and used by Gaffke and Pukelsheim (2008a), Gaffke and Pukelsheim (2008b) to model biproportionality.

$$\begin{aligned}
\text{minimize} \quad & \sum_{e \in E(\mathcal{V})} \sum_{t=1}^{H} y_e^t \log \left(\frac{t}{\mathcal{V}_e} \right) && && (10)\\
\text{subject to} \quad & \sum_{t=1}^{H} y_e^t = x_e && \text{for every } e \in E(\mathcal{V}), && (11)\\
& \sum_{\substack{e \in E(\mathcal{V}):\\ e_i = v}} x_e = m_v && \text{for every } i \in \{1, \ldots, d\} \text{ and } v \in N_i, && (12)\\
& x_e \geq I_e && \text{for every } e \in E(\mathcal{V}), && (13)\\
& x_e \leq U_e && \text{for every } e \in E(\mathcal{V}), && (14)\\
& y_e^t \in \{0, 1\} && \text{for every } e \in E(\mathcal{V}) \text{ and } t \in \{1, \ldots, H\}. && (15)
\end{aligned}$$

For each $e \in E(\mathcal{V})$ and $t \in \{1, \ldots, H\}$ we have a binary variable y_e^t whose cost in the objective function is given by $\log(t/\mathcal{V}_e)$. The variable x_e represents the total number of seats to be allocated in the apportionment for candidates of tuple e; constraint (11) takes care of aggregating the seats in these variables. Constraint (12) enforces every feasible solution to satisfy the marginals and constraints (13) and (14) ensure respecting the bounds. The following theorem states the main result for this setting with plurality.

Theorem 2 *Let $\mathcal{V}$, m, H, I, and U be such that the linear relaxation of (10)-(15) is feasible. Let $\alpha_1, \ldots \alpha_d$ be nonnegative integers such that $\sum_{i=1}^{d} 1/(\alpha_i + 2) \leq 1$. Then, there exists an α-approximate d-dimensional proportional apportionment.*

As mentioned above, this theorem extends that of Cembrano et al. (2022, Theorem 4), which would correspond to the same statement if we omit constraints (13) and (14). The discrepancy approach (Cembrano et al. 2022, Algorithm 1 and Theorem 5) remains valid for this setting as well, so the only result we need to extend is their Lemma 5, stating that any rounding of an optimal solution of the linear relaxation (without constraints (13) and (14)) satisfies the proportionality condition. In our context, this entails showing that any rounding of an optimal solution of the linear relaxation of (10)-(15) satisfies the modified proportionality condition (9). In order to do so, we first observe that by duality, for any feasible solution (x, y) of the linear relaxation of (10)–(15), we have that (x, y) is optimal if and only if there exists a dual solution $(\Lambda, \omega^-, \omega^+, \beta)$ such that the following conditions hold:

$$\sum_{i=1}^{d} \Lambda_{e_i} + \omega_e^- + \omega_e^+ + \beta_e^t - \log\left(\frac{t}{\mathcal{V}_e}\right) \leq 0 \quad \text{for all } e \in E(\mathcal{V}) \text{ and } t \in \{1, \ldots, H\}, \tag{16}$$

$$y_e^t \left[\sum_{i=1}^{d} \Lambda_{e_i} + \omega_e^- + \omega_e^+ + \beta_e^t - \log\left(\frac{t}{\mathcal{V}_e}\right)\right] = 0 \quad \text{for all } e \in E(\mathcal{V}) \text{ and } t \in \{1, \ldots, H\}, \tag{17}$$

$$\omega_e^- \left(\sum_{t=1}^{H} y_e^t - I_e\right) = 0 \quad \text{for all } e \in E(\mathcal{V}), \tag{18}$$

$$\omega_e^- \geq 0 \quad \text{for all } e \in E(\mathcal{V}), \tag{19}$$

$$\omega_e^+ \left(U_e - \sum_{t=1}^{H} y_e^t\right) = 0 \quad \text{for all } e \in E(\mathcal{V}), \tag{20}$$

$$\omega_e^+ \leq 0 \quad \text{for all } e \in E(\mathcal{V}), \tag{21}$$

$$\beta_e^t (y_e^t - 1) = 0 \quad \text{for all } e \in E(\mathcal{V}) \text{ and } t \in \{1, \ldots, H\}, \tag{22}$$

$$\beta_e^t \leq 0 \quad \text{for all } e \in E(\mathcal{V}) \text{ and } t \in \{1, \ldots, H\}, \tag{23}$$

where Λ_v is the dual variable associated to the constraint (12) for every $i \in \{1, \ldots, d\}$ and $v \in N_i$, ω_e^- is the dual variable associated to the constraint (13) for every e, ω_e^+ is the dual variable associated to the constraint (14) for every e, and β_e^t is the dual variable associated to the upper bound of one on the value of y_e^t for every $e \in E(\mathcal{V})$ and $t \in \{1, \ldots, H\}$.

We refer to a tuple $(x, y, \Lambda, \omega^-, \omega^+, \beta)$ jointly satisfying (16)-(23), with (x, y) feasible for the linear relaxation of (10)–(15), as an optimal primal-dual pair for this linear relaxation. The following lemma establishes that any rounding of an optimal solution x of this linear relaxation satisfies the proportionality condition (9). As mentioned, this lemma along with the results of Cembrano et al. (2022, Theorem 1, Theorem 5) allows us to conclude Theorem 2. We remark that when $d = 2$, the feasibility of the linear relaxation is guaranteed when the bounds I and U are consistent with the marginals m, i.e., when condition (4) holds. This can be seen by modeling the apportionment as a network flow (Gaffke and Pukelsheim 2008a) and applying Hoffman's Circulation Theorem (Hoffman 1960). For $d \geq 3$, it is guaranteed when $I_e = 0$, $U_e = H$, and all entries of $\mathcal{V}$ are strictly positive (Cembrano et al. 2022).

Lemma 2 *Let $(x, y, \Lambda, \omega^-, \omega^+, \beta)$ be an optimal primal-dual pair for the linear relaxation of (10)-(15) with input $\mathcal{V},\ m,\ H,\ I$, and U. If $\bar{x}$ is an integral vector with entries in $E(\mathcal{V})$ such that $\bar{x}_e \in \{\lfloor x_e \rfloor, \lceil x_e \rceil\}$ for every $e \in E(\mathcal{V})$, then defining $\lambda_v = \exp(\Lambda_v)$ for every $i \in \{1, \dots, d\}$ and $v \in N_i$ we have that*

$$\bar{x}_e \in mid\left(\left[\!\left[\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}\right]\!\right] \cup \{I_e, U_e\}\right) \qquad \textit{for every } e \in E(\mathcal{V}).$$

The proof of the lemma follows a similar structure to the proof of Lemma 5 by Cembrano et al. (2022). However, the additional constraints of the linear relaxation and the modified proportionality condition require a more careful analysis. Specifically, we first need to consider the case in which a certain entry of x is equal to its lower or upper bound, and show that this value belongs to the set $\text{mid}\left(\left[\!\left[\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}\right]\!\right] \cup \{I_e, U_e\}\right)$. When this is not the case, we apply the optimality conditions appropriately to show that the scaled entry $\mathcal{V}_e \exp(\sum_{i=1}^{d} \Lambda_{e_i})$ is equal to $\lceil x_e \rceil$ and conclude that any rounding of x_e corresponds to the median value.

Proof Let $(x, y, \Lambda, \omega^-, \omega^+, \beta)$, λ, and $\bar{x}$ be as in the statement. For the first part of the proof, we follow Cembrano et al. (2022). For every $e \in E(\mathcal{V})$, we consider the value

$$t_e = \max\{t \in \{1, \dots, H\} : y_e^t > 0\}$$

when $x_e > 0$ and $t_e = 0$ when $x_e = 0$, and we claim that for every $e \in E(\mathcal{V})$ it holds that $t_e = \lceil x_e \rceil$. This is straightforward for the tuples $e \in E(\mathcal{V})$ with $x_e = 0$. Constraint (11) further implies that $t_e \geq \lceil x_e \rceil$. Suppose towards a contradiction that $e \in E(\mathcal{V})$ is such that $t_e > \lceil x_e \rceil$. Then, there are at least two fractional values in $y_e^1 \dots, y_e^H$; consider in addition the solution (x, Y) constructed as follows:

$$Y_e^t = \begin{cases} 1 & \text{for every } t < \lfloor x_e \rfloor + 1, \\ x_e - \lfloor x_e \rfloor & \text{for } t = \lfloor x_e \rfloor + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The solution (x, Y) is feasible for the linear relaxation of (10)-(15) since

$$\sum_{t=1}^{H} Y_e^t = (\lfloor x_e \rfloor + 1) - 1 + x_e - \lfloor x_e \rfloor = x_e.$$

Additionally, since for every $e \in E(\mathcal{V})$ the function $\log(t/\mathcal{V}_e)$ is strictly increasing as a function of $t \in \{1, \ldots, H\}$, we obtain that $\sum_{t=1}^{H} Y_e^t \log(t/\mathcal{V}_e) < \sum_{t=1}^{H} y_e^t \log(t/\mathcal{V}_e)$, a contradiction to the optimality of (x, y). This concludes the proof of the claim.

We now let $e \in E(\mathcal{V})$ be arbitrary and note that since x satisfies constraint (13) we have that $I_e \le x_e \le U_e$. In particular, if $I_e = U_e$ the condition in the statement holds trivially. We thus consider an arbitrary tuple $e \in E(\mathcal{V})$ with $I_e < U_e$ and distinguish three cases; we will conclude the condition in the statement for each of them.

If $x_e = I_e$, from the fact that $t_e = \lceil x_e \rceil$ we have that $y_e^{I_e+1} = 0$ and then condition (22) implies that $\beta_e^{I_e+1} = 0$. Moreover, since $I_e < U_e$ we have that $x_e < U_e$ and condition (20) implies that $w_e^+ = 0$. Replacing in condition (16) we obtain

$$\log\left(\frac{I_e + 1}{\mathcal{V}_e}\right) - \sum_{i=1}^{d} \Lambda_{e_i} \ge \omega_e^-.$$

Since $\omega_e^- \ge 0$ from (19), this implies that $\mathcal{V}_e \exp(\sum_{i=1}^{d} \Lambda_{e_i}) \le I_e + 1$. Therefore, we have either $\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i} < I_e$ or $I_e \in \left[\!\left[\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}\right]\!\right]$. In both cases, we conclude that

$$\bar{x}_e = x_e = I_e \in \operatorname{mid}\left(\left[\!\left[\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}\right]\!\right] \cup \{I_e, U_e\}\right).$$

If $x_e = U_e$, from the fact that $t_e = \lceil x_e \rceil$ we have that $y_e^{U_e} = 1$ and then condition (17) yields

$$\sum_{i=1}^{d} \Lambda_{e_i} + \omega_e^- + \omega_e^+ + \beta_e^{U_e} - \log\left(\frac{U_e}{\mathcal{V}_e}\right) = 0.$$

Moreover, since $I_e < U_e$ we have that $x_e > I_e$ and condition (18) implies that $\omega_e^- = 0$. Replacing in the previous equation, we obtain

$$\log\left(\frac{U_e}{\mathcal{V}_e}\right) - \sum_{i=1}^{d} \Lambda_{e_i} = \omega_e^+ + \beta_e^{U_e} \le 0,$$

where we also used that $\omega_e^+ \le 0$ and $\beta_e^{U_e} \le 0$ due to conditions (21) and (23), respectively. This implies that $\mathcal{V}_e \exp(\sum_{i=1}^{d} \Lambda_{e_i}) \ge U_e$. Therefore, we have either $\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i} > U_e + 1$ or $U_e \in \left[\!\left[\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}\right]\!\right]$. In both cases, we conclude that

$$\bar{x}_e = x_e = U_e \in \operatorname{mid}\left(\left[\!\left[\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}\right]\!\right] \cup \{I_e, U_e\}\right).$$

Finally, when $I_e < x_e < U_e$, from conditions (18) and (20) we have that $\omega_e^- = \omega_e^+ = 0$. Fixing $h = t_e = \lceil x_e \rceil$, the complementary slackness condition (17) implies that

$$\sum_{i=1}^{d} \Lambda_{e_i} + \beta_e^h - \log\left(\frac{h}{\mathcal{V}_e}\right) = 0, \tag{24}$$

since $y_e^h > 0$. By (23) we have that $\beta_e^h \leq 0$ and therefore we conclude that $\sum_{i=1}^{d} \Lambda_{e_i} \geq \log(h/\mathcal{V}_e)$, implying $t_e = h \leq \mathcal{V}_e \exp(\sum_{i=1}^{d} \Lambda_{e_i})$. We know that for every $t > t_e$ it holds $y_e^t = 0$, so the complementary slackness condition (22) implies that $\beta_e^t = 0$. Therefore, condition (16) implies that $\sum_{i=1}^{d} \Lambda_{e_i} \leq \log(t/\mathcal{V}_e)$, which is satisfied in particular for $t = t_e + 1$. We conclude that $\mathcal{V}_e \exp(\sum_{i=1}^{d} \Lambda_{e_i}) \leq t_e + 1$. Putting it all together, we have that

$$\lceil x_e \rceil \leq \mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i} \leq \lceil x_e \rceil + 1,$$

implying $\lceil x_e \rceil \in [\![\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}]\!]$. If x_e is integer, then $\bar{x}_e = x_e = \lceil x_e \rceil \in \{I_e + 1, \ldots, U_e - 1\}$, so we conclude. If x_e is fractional, we have $0 < y_e^h < 1$ for $h = t_e$, thus the complementary slackness condition (22) implies that $\beta_e^h = 0$. Therefore, we conclude from (24) that $\mathcal{V}_e \exp(\sum_{i=1}^{d} \Lambda_{e_i}) = h = t_e = \lceil x_e \rceil$. When $\bar{x}_e = \lceil x_e \rceil$ we have that

$$\bar{x}_e = \lceil x_e \rceil = \mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i} \leq \lceil x_e \rceil + 1 = \bar{x}_e + 1.$$

Similarly, when $\bar{x}_e = \lfloor x_e \rfloor$ we have that

$$\bar{x}_e + 1 = \lfloor x_e \rfloor + 1 = \lceil x_e \rceil = \mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i} \geq \lfloor x_e \rfloor = \bar{x}_e.$$

In both cases we have $\bar{x}_e \in [\![\mathcal{V}_e \prod_{i=1}^{d} \lambda_{e_i}]\!]$, thus we conclude once again. □

4 Description of the apportionment methods

In the application we consider, there is a set of districts D, a set of lists L, and a set of genders G. There is also a set of candidates $\mathcal{C}$, and each candidate $c \in \mathcal{C}$ competes in a district $\mathbf{d}(c) \in D$, belongs to a list $\boldsymbol{\ell}(c) \in L$, and is of gender $\mathbf{g}(c) \in G$. The results of the election consist of the votes $v(c) \in \mathbb{N}$ each candidate c obtains. We are also given a vector $q \in \mathbb{N}^D$ containing the number of seats to be assigned in each district $d \in D$ according to the law, and a strictly positive integer number H corresponding to the total number of seats to be allocated. Since we are interested in the aggregated votes when defining algorithms and evaluating their performance, we construct a matrix $\mathcal{V} \in \mathbb{N}^{D\times L\times G}$ such that $\mathcal{V}_{d\ell g}$ is the sum of the votes of the candidates of list ℓ and

Table 1 Votes disaggregated by districts, lists, and genders in our example used to illustrate the methods. Lists A to D have candidates of both genders in all districts, while lists E and F each compete in a single district

Female	D1	D2	D3	Male	D1	D2	D3
A	583,494	365,796	364,104	**A**	242,112	484,302	145,398
B	61,674	48,078	43,416	**B**	20,454	18,762	23,238
C	192,546	431,472	857,646	**C**	231,600	461,640	452,682
D	110,664	227,484	39,774	**D**	65,268	72,696	36,060
E	0	0	0	**E**	160,686	0	0
F	0	0	74,064	**F**	0	0	99,618

gender g in district d, i.e.

$$\mathcal{V}_{d\ell g} = \sum_{\substack{c\in\mathcal{C}:\, \mathbf{d}(c)=d,\\ \boldsymbol{\ell}(c)=\ell,\, \mathbf{g}(c)=g}} v(c).$$

The objective of the methods discussed in this work is to find an assignment of candidates to seats represented by a function χ, so $\chi(c) = 1$ if candidate c is elected and $\chi(c) = 0$ otherwise. We naturally require that $\sum_{c\in\mathcal{C}} \chi(c) = H$.

To illustrate how our methods work, we use a small example of an artificial election consisting of six lists from A to F, three districts from D1 to D3, and a house size of $H = 33$ seats.[7] Table 1 shows the votes disaggregated in all three dimensions.

4.1 The Chilean constitutional convention method (CCM)

We start by describing the method used in the recent Chilean Constitutional Convention election on May 15–16, 2021. In this method, the seats of each district $d \in D$ are distributed across the lists according to their votes using the (1-dimensional) Jefferson/D'Hondt method, i.e., we compute a vector

$$r^d \in \mathcal{A}^1(\mathcal{V}_{d,\cdot,+}, q_d)$$

for each $d \in D$. A second distribution of the r^d_ℓ seats assigned to each list in each district among its sublists is then performed through a new round of the Jefferson/D'Hondt method, and the corresponding number of top-voted candidates of each sublist is provisionally selected; if at this point the elected candidates fulfill gender balance,[8] the seats are assigned. Otherwise, the elected candidate with the smallest number of votes among those of the over-represented gender is replaced by the non-elected candidate with the most votes among those of the other gender and the same sublist.

[7] This data is actually a modification of the results of the 2021 Constitutional Convention election in Chile, where only the six most-voted lists in districts 10, 11, and 12 were considered with modified district marginals, in an attempt to illustrate the coincidences and differences between methods observed with the full data.

[8] Gender balance in this context refers to the same number of candidates of each gender if the number of seats of the district is even and one more man/woman otherwise.

This replacement procedure is repeated until gender balance is achieved. We refer to the Chilean electoral laws (Diario Oficial de la República de Chile: 2015; Diario Oficial de la República de Chile 2020) for the legal description of the method, and to Mathieu and Verdugo (2024) for an algorithmic and axiomatic analysis.

4.2 The 3-proportional method (TPM)

The 3-proportional method directly applies the notion of multidimensional proportionality introduced in Sect. 2 for the case $d = 3$ with $N_1 = D$, $N_2 = L$ and $N_3 = G$. As described there, a 3-proportional apportionment does not always exist, but an α-approximate 3-proportional apportionment is guaranteed to exist as long as $\sum_{i=1}^{3} 1/(\alpha_i + 2) \leq 1$. In particular, this holds when $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 4$.

In order to properly define the method, we need to define the marginals. For the case of districts, we naturally set $m_d = q_d$ for each $d \in D$. For each list $\ell \in L$, we let m_ℓ be the number of seats corresponding to the total number of votes that the list gets in the country according to the Jefferson/D'Hondt method, i.e., $m_\ell = (\mathcal{A}^1(\mathcal{V}_{+,\cdot,+}, H))_\ell$. For each gender, we aim to assign half of the seats; when H is odd we simply break the tie in favor of the gender obtaining the highest number of votes:

$$m_g = \begin{cases} \left\lceil \frac{H}{2} \right\rceil & \text{if } \mathcal{V}_{+,+,g} > \frac{1}{2}\mathcal{V}_{+,+,+}, \\ \left\lfloor \frac{H}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

We then aim to find a 3-proportional apportionment with no deviation from the districts and list marginals and the least possible deviation from gender marginals, i.e., we set $\alpha_1 = \alpha_2 = 0$ and

$$\alpha_3 = \min\{z \in \{0, \dots, 4\} : \mathcal{A}^3(\mathcal{V}, m, H; (0, 0, z)) \neq \emptyset\},$$

and we naturally consider $x = \mathcal{A}^3(\mathcal{V}, m, H; \alpha)$, where we denote $(\alpha_1, \alpha_2, \alpha_3)$ by α.. In case $\mathcal{A}^3(\mathcal{V}, m, H; (0, 0, z)) = \emptyset$ for every $z \in \{0, \dots, 4\}$, we allow a deviation of 1 with respect to district marginals, i.e., we fix $\alpha_1 = 1$, $\alpha_2 = 0$, and

$$\alpha_3 = \min\{z \in \{0, \dots, 4\} : \mathcal{A}^3(\mathcal{V}, m, H; (1, 0, z)) \neq \emptyset\},$$

and we return $x = \mathcal{A}^3(\mathcal{V}, m, H; \alpha)$. The correctness of this procedure is guaranteed by the aforementioned existence result by Cembrano et al. (2022). Once the seats $x_{d\ell g}$ for each tuple $(d, \ell, g) \in D \times L \times G$ are fixed, they are distributed among the sublists, as before, through a new round of the (1-dimensional) Jefferson/D'Hondt method. The corresponding number of top-voted candidates of each district, sublist, and gender is then selected.

The result of the described method is shown in Table 2.

This solution is actually found with deviations $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so if we sum the apportionment across gender and lists we obtain the corresponding district magnitudes of 6, 15, and 12; the gender allocation respects the rule with 17 and 16 seats for women

Table 2 Number of seats assigned to candidates of each tuple given by a district, a list, and a gender in the 3-proportional method described in Sect. 4.2, for the example election described in Table 1

Female	D1	D2	D3	Male	D1	D2	D3
A	3	2	2	**A**	1	4	1
B	1	0	0	**B**	0	0	0
C	0	2	5	**C**	1	4	3
D	0	2	0	**D**	0	1	0
E	0	0	0	**E**	0	0	0
F	0	0	0	**F**	0	0	1

and men; and the list allocation is the one given by the Jefferson/D'Hondt method with a multiplier $\lambda = 6 \cdot 10^{-6}$.

4.3 The 3-proportional method with threshold (TPM3)

In this method, we incorporate a threshold on the percentage of votes obtained by a list in order to be eligible for the apportionment on top of the 3-proportional method. More specifically, we only include in the process the set of lists $\ell \in L$ that obtain at least a 3% of the votes. In order to implement this, we set $U_\ell = 0$ for every $\ell \in L$ with $\mathcal{V}_{+,\ell,+} < 0.03\mathcal{V}_{+,+,+}$, $U_\ell = H$ for every other list ℓ, and $I_\ell = 0$ for every $\ell \in L$. We then set $m_\ell = (\mathcal{A}^1(\mathcal{V}_{+,\cdot,+}, H, I, U))_\ell$ and run the rest of the method in the same way as TPM.

4.4 The 3-proportional method with plurality (TPP)

In this method, we modify the 3-proportional method to ensure that the top-voted candidate of each district is elected. The procedure is identical to TPM except that now we add $I, U \in \mathbb{N}^{D\times L\times G}$ as arguments when applying $\mathcal{A}^3$. Letting

$$c^*(d) = \arg \max_{c\in\mathcal{C}:\mathbf{d}(c)=d} v(c)$$

denote the top-voted candidate[9] of district d for each $d \in D$, I is defined as

$$I_{d\ell g} = \begin{cases} 1 & \text{if } \boldsymbol{\ell}(c^*(d)) = \ell \text{ and } \mathbf{g}(c^*(d)) = g, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for every } (d, \ell, g) \in D \times L \times G,$$

and $U_{d\ell g} = H$ for every $(d, \ell, g) \in D \times L \times G$. This modification guarantees that $c^*(d)$ is elected for every $d \in D$.

[9] We have assumed for simplicity that there is a unique top-voted candidate in each district; an arbitrary tie-breaking rule can be used otherwise.

Table 3 Comparison of seat allocation by list under the methods described in Sects. 4.1–4.5, for the example election described in Table 1

	CCM	TPM	TPM3	TPP	TPP3
A	13	13	13	12	13
B	0	1	1	1	1
C	17	15	16	15	15
D	2	3	3	3	3
E	0	0	0	1	1
F	1	1	0	1	0

4.5 The 3-proportional method with threshold and plurality (TPP3)

In this method, we ensure that the top-voted candidate of each district is elected as in TPP, and we consider the threshold condition as in TPM3. In order to implement this, we proceed in the same way as described in Sect. 4.4, except that we first compute the list marginals in order to respect the threshold condition, ensuring that each list below the threshold gets precisely its amount of top-voted candidates. Specifically, we define $I_{d\ell g}$ and $U_{d\ell g}$ as in Sect. 4.4, we set $U'_\ell = \sum_{d\in D}\sum_{g\in G} I_{d\ell g}$ for every $\ell \in L$ with $\mathcal{V}_{+,\ell,+} < 0.03 \cdot \mathcal{V}_{+,+,+}$, $U'_\ell = H$ for every other list ℓ, and $I'_\ell = 0$ for every $\ell \in L$. We then set $m_\ell = (\mathcal{A}^1(\mathcal{V}_{+,\cdot,+}, H, I', U'))_\ell$ and run the rest of the method in the same way as TPP.

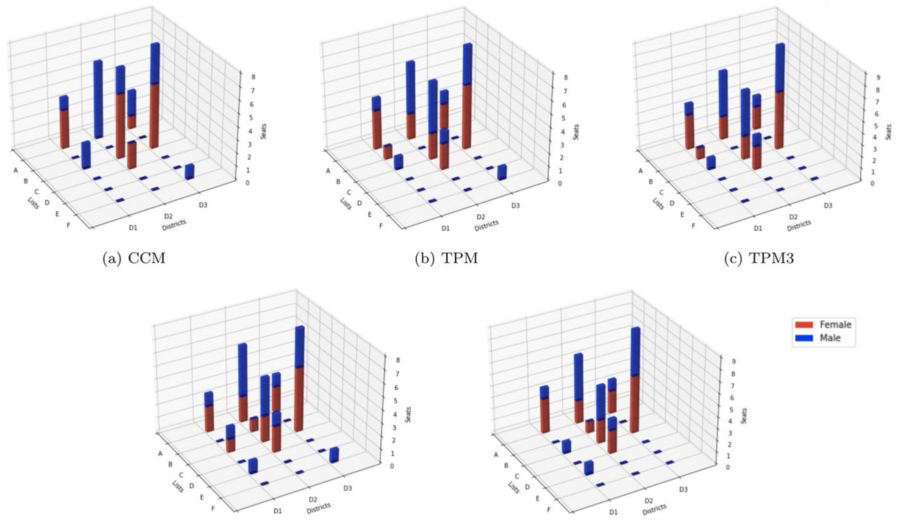
Comparison summary. Table 3 summarizes the results for each method, and Fig. 1 shows them in further detail, with the corresponding number of seats by list, district, and gender. The figure shows how the 3-proportional methods achieve gender parity only across districts and lists, while the constraint is active in each district in the case of CCM. We can also observe how the threshold prevents list F from having a seat that was originally allocated by proportionality. List E does not have enough votes to obtain representation in TPM but gets a seat in both TPP and TPP3 due to having the most-voted candidate in district 1.[10]

5 Experimental results and analysis

In this section, we study the outcomes of the methods described above with the data from the 2021 Chilean Constitutional Convention election. We evaluate their performance in terms of the proportionality of the results, the representativeness of the elected candidates, and the value of each cast vote. We also use the election data to test the deviations from the marginals required by the methods based on multidimensional proportionality.

We first give some context for the Chilean Constitutional Convention election. While 155 seats were to be allocated in total, 17 of these seats were reserved for ethnic minority groups and assigned through a parallel election; our analysis thus focuses on the 138 non-ethnic seats. The country is divided into 28 electoral districts, each

[10] Observe that this happens despite the list getting less than 3% of the votes as well, as the method fixes the number of seats for the lists with less than 3% of the votes to their number of top-voted candidates.



(a) CCM (b) TPM (c) TPM3

(d) TPP (e) TPP3

Fig. 1 Comparison of the apportionment of seats by lists, districts, and gender under the methods described in Sects. 4.1–4.5, for the example election described in Table 1. Each bar height represents the number of seats obtained by a list in a certain district, segmented by the number of seats corresponding to men (blue) and women (red)

receiving between 3 and 8 seats. Candidates run in open lists, and each voter votes for at most one candidate.

A total of 70 lists and over 1300 candidates competed in the election. Among them, only 20 lists and two independent candidates[11] obtained enough votes to be elected in any of the methods described in Sect. 4. For exposition purposes, in Sects. 5.1 and 5.2 we omit the other lists and independent candidates, which does not affect our results.[12] Among the main political alliances, we encounter the XP list, which encompasses traditional and newer right-wing parties, the YB list, formed by the center-left parties that governed Chile between 1990 and 2010, and the YQ list, which contains left-wing parties. In addition, two conglomerates of lists containing politically independent candidates obtained considerable support; following their campaign, we denote them by LP (for *Lista del Pueblo*) and INN (for *Independientes No Neutrales*).[13] Besides these two lists, we use the election codes to represent the lists.

We remark that a deviation of one seat from the gender marginals was needed when running TPM3 and TPP3 on the electoral data, while no deviation was needed in the case of TPM and TPP. In Sect. 5.4, we will see that this is no coincidence, as deviations needed in practice tend to be much smaller than the worst-case guarantees.

[11] These candidates are denoted as IND1 and IND9 (according to the districts in which they run).

[12] None of these lists and independent candidates obtained more than 0.51% of the votes, and they jointly represent less than the 10% of the total votes.

[13] This grouping is standard; see, e.g., https://2021.decidechile.cl/#/ev/2021/ct/2021.N/. Full election data is available on the website of Servicio Electoral de Chile (2021).

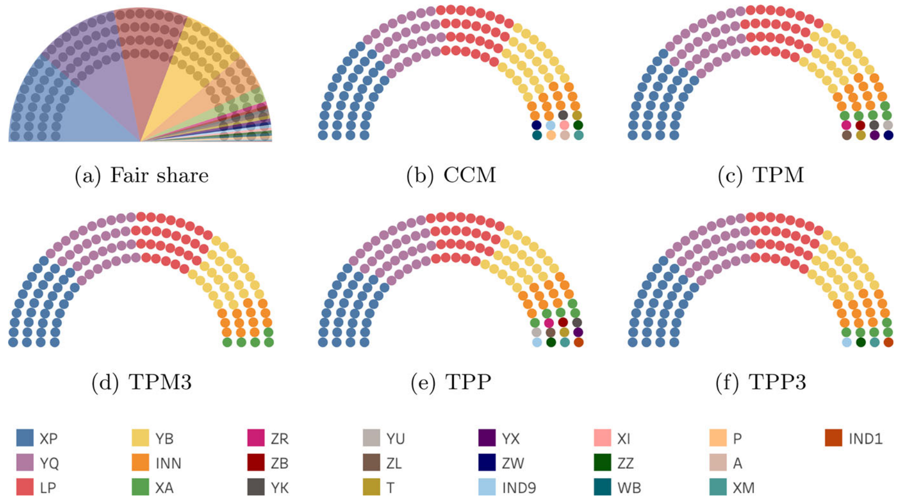


Fig. 2 Political representation obtained by list under each method and fair share apportionment for comparison

5.1 Proportionality

Proportionality in the election results is measured through the difference between the induced political distribution and the (d-dimensional) fair share, which was introduced in Sect. 2 and constitutes a natural notion of perfectly fair distribution. Figure 2 provides a graphical view of the political representation obtained by the lists under each method and this perfectly fair political distribution. The data used for this figure is contained in Table 8 in Appendix A.

It is observed that the global methods, especially those without a threshold (TPM and TPP), produce a political distribution much closer to the fair share than the local method CCM. In particular, the most-voted list XP is not as overrepresented as in CCM, and seats are assigned to a subset of the top-voted lists as close to proportionality as possible. As an illustrative example, one may observe that the list XA—the sixth most-voted one with almost 4% of the votes—receives five seats under the global methods and no seat under CCM. The result under TPM and TPP is a highly varied convention made up of large political blocks together with multiple lists that obtain a single representative. While TPM3 considerably reduces the number of lists entering the convention, the allocation produced by TPP3 stands as a midpoint between those produced by TPM3 and TPP, leaving out lists with too few votes at a national level but allowing the representation of strong local projects. In particular, the XM list, which is a project of the southernmost part of Chile (Magallanes), as well as the independents from districts 1 and 9, which were the first majorities of their districts, enter the convention.

The deviation from the fair share can be naturally quantified through the Euclidean distance between a specific apportionment and the fair share, a value called *Gallagher Index* in the literature. This deviation can be measured either globally in the country or

Table 4 Gallagher Index (%) computed for the global political distribution, the average of the local political distributions, and the 3-dimensional seat distribution

	CCM	TPM	TPM3	TPP	TPP3
Global GI	4.6	1.8	3.8	1.3	2.9
Local GI avg.	14.5	18.5	19.0	18.2	18.5
3-dim GI	3.9	3.6	3.8	3.6	3.8

locally within each district (taking the average thereof). We call the respective values *global Gallagher Index* and *local Gallagher Index*. Furthermore, we can define a 3-dimensional Gallagher Index in order to evaluate the proportional allocation of seats across districts, lists, and genders simultaneously: We take the Euclidean distance between the apportionment and the 3-dimensional fair share defined in Sect. 2.

Table 4 shows the described indices for each method.[14]

Naturally, the district averages show that the local method (CCM) is closer to the local fair share than global methods, as the former is designed to pursue local proportionality. However, the local errors produced by CCM add up across districts, leading to a considerably larger global Gallagher Index than global methods—particularly than TPM and TPP—and a slightly larger 3-dimensional index. This is again natural, as TPM and TPP compute the seats to be allocated to each list in a global manner, thus leading to a political distribution that is closer to proportionality on a national level. In fact, TPP ends up with the lowest global deviation, followed by the TPM, while TPP3 again stands as a midpoint between the methods with and without a threshold in this regard. In terms of the 3-dimensional index, the results are similar between all the methods, with differences of at most 0.3 percentage points. This similarity can be explained by the fact that the 3-proportional methods generate a good allocation to lists but produce local distortions. Conversely, the national allocation to lists does not adjust correctly to the votes in CCM but the district allocation does, highlighting a fundamental trade-off between local and global political representation.

5.2 Representativeness

In this section, we analyze the average of votes and percentages obtained by the elected candidates under each method. These values are summarized in Table 5. We additionally include, as a reference for the maximum attainable representativeness given the constraints of the election, an apportionment method that chooses the most-voted candidates constrained to district marginals and global gender parity. We denote this method by *Greedy*.

By construction, all methods that involve correction mechanisms in order to ensure gender parity imply a certain degree of loss of votes, since candidates who may have been elected without corrections are substituted by other non-elected ones with a lower number of votes. A relevant observation is the fact that TPP and TPP3 achieve a higher average than CCM, despite the greedy nature of the latter: It just replaces candidates when necessary and in a local manner. One explanation is the presence of locally top-voted candidates who are not elected in other methods and obtain a

[14] The Gallagher Index computed by district can be found in Table 9 in Appendix A.

Table 5 Average votes obtained by the elected candidates under each method, and the average percentage of votes obtained by elected candidates with respect to votes cast in their districts

	CCM	TPM	TPM3	TPP	TPP3	Greedy
Avg. votes	14,126	14,048	13,848	14,406	14,198	15,227
Avg. district %	31.2%	31.1%	30.7%	32.1%	31.7%	33.6%

Fig. 3 Vote power by district as a ratio of global vote power for a local method such as CCM. Here, the voting power is defined as the ratio between the number of seats assigned to the district and the number of votes cast in the district

large number of votes. This is particularly relevant since, in addition to the property of plurality—and a representation threshold in the case of TPP3—these methods obtain the best results in terms of representation, followed by CCM and TPM. It is observed that the threshold decreases the average votes in this case. This behavior, however, is not a direct consequence of its application but rather depends on the instance. In this particular case, the negative effect on candidates of small lists who were not elected due to this threshold was more important than the positive effect on candidates of bigger lists.

5.3 One person, one vote?

In this section, we compare the value of a cast vote under each method in terms of political impact.[15] Under local methods (CCM), each vote counts only for electing the seats assigned to the corresponding district, and therefore its power can be measured by the ratio between the number of seats assigned to the district and the number of votes cast in the district. Figure 3 shows this indicator for each district, divided by the overall ratio between the house size and votes to normalize. We observe that the votes of people living in central districts are less powerful, as defined previously, than the votes of people living in extreme districts, reaching a factor of up to 5.93 between them. Conversely, global methods without plurality ensure that, in terms of political representation, every vote is equally valuable as seats are assigned to political lists proportionally to the number of votes they receive. Global methods with plurality combine both features—global representation and election of locally top-voted candidates—so each vote is valuable in terms of national political distribution but has special relevance for the district where it is cast.

[15] Unlike the other subsections, in this one, the votes of the lists without enough votes to obtain a seat under some method are considered, as they are relevant for the measure.

This is related to the aforementioned trade-off between local and global representation. Incorporating the different criteria as additional dimensions, rather than running separate elections, may be a reasonable way to get closer to the widely-known principle of *one person, one vote*.

5.4 Deviations in the number of seats

We finally conduct simulations to assess the deviations from the marginals that the proposed methods require in practice. Our simulations consider four vote models with different distributions and the same instance size as the original election (28 districts, 22 lists, 2 genders, and a house size of 138 seats). Denoting by $\mu_{\mathcal{V}}$ the mean of the votes $\mathcal{V}_{d\ell g}$ and $\mathcal{V}^n_{d\ell g}$ the n-th random sample from the distribution F, we consider the following models:

1. $\mathcal{V}^n_{d\ell g} = \mathcal{V}_{d\ell g} \cdot X_n$, where X is drawn from Normal$(1, 0.1)$, and $\mathcal{V}_{d\ell g}$ are the original votes;
2. $\mathcal{V}^n_{d\ell g}$ is drawn from Poisson$(\theta_{d\ell})$, and $\theta_{d\ell} \sim$ Gamma$(1, \mu_{\mathcal{V}})$;
3. $\mathcal{V}^n_{d\ell g}$ is drawn from Uniform$(1, 2\mu_{\mathcal{V}})$;
4. $\mathcal{V}^n_{d\ell g}$ is drawn from Pareto$(0.5, 1000)$.

The first instance aims to use a normal perturbation of the original votes. The second distribution emulates an election with heterogeneous preferences for lists across different districts, governed by the parameters $\theta_{d\ell}$. In the third and fourth models, we consider equally distributed entries, sampled from a uniform or a heavy-tailed Pareto distribution, respectively. Note that the number of votes and the number of seats per district may be too inconsistent in the last three models; to correct this, we scale each entry (d, ℓ, g) of each generated matrix by $\frac{P_d}{P_+} \cdot \frac{\mathcal{V}_{+++}}{\mathcal{V}_{d++}}$, where P_d is the population of district d and P_+ the total population (according to the last official census in 2017). We set the simulations to aim for a similar mean to the original vote matrix ($\mu_{\mathcal{V}} \approx 4200$). Model 4 does not have a finite mean, but the median is similar.

In Fig. 4 we can observe that, despite the upper bound of $\alpha = (1, 0, 4)$ on the deviations needed to find an approximate 3-proportional apportionment, no deviation is needed in most simulations to find an output of our methods. When this is not the case, no deviation from the district marginals and a deviation of only one seat—or very rarely, two or three seats—from the gender marginals is required. The vote instances drawn from the uniform and Pareto distributions tend to perform slightly worse in this sense, which can be understood as a difficulty in finding multidimensional proportional apportionments when the preferences for lists and genders are distributed similarly overall but inconsistently across districts.

5.5 Results for the 2023 parliamentary election in Finland

For the sake of replicability, we conduct the same simulations and evaluation procedures described throughout the previous sections in the context of Finland's 2023

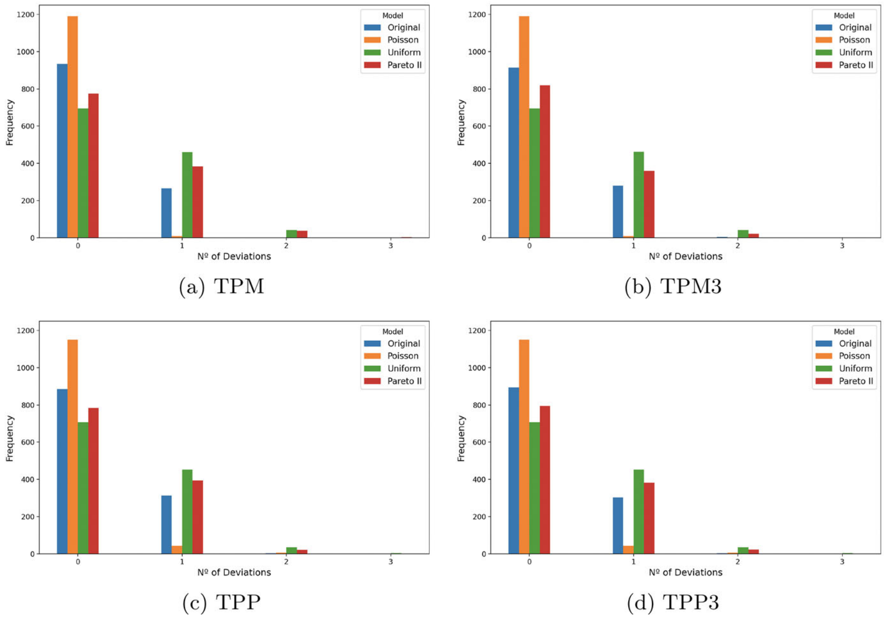


Fig. 4 Distribution of absolute deviation magnitude in gender marginals for the 3-proportional methods. No deviations in other marginals were ever required. We conduct a simulation of $N = 1200$ random vote instances from 4 different distributions and compute the frequency of the deviations for each method and instance

parliamentary election.[16] In every district, candidates can run as independent candidates or as part of a political party, and multiple political parties can run as a joint list as part of an electoral agreement. Similar to Chile, the relevant political dimension is the *list* (which can consist of single or multiple parties), and elected candidates are chosen via the Jefferson/D'Hondt method district-wise. However, in Finnish elections, the seats obtained by a list are directly assigned to the most-voted candidates, unlike the Chilean case, where the Jefferson/D'Hondt method is applied again if a list consists of multiple parties. Additionally, Finland does not use any *ex-post* nor *ex-ante* gender parity mechanism. Some parties voluntarily propose lists with equal numbers of male and female candidates in alternating order, as a political signal.

While the Chilean election had 1278 candidates, a house size of 138 seats, and 5.7 million votes were cast in 28 districts with magnitudes ranging from 3 to 8, in the Finnish election there were 2424 candidates, a house size of 200 seats, and 3.1 million votes were cast in 13 districts of magnitudes ranging from 1 to 37. We denote the original election results by the code E23, and we include the CCM method in the simulation for a full comparison. We now discuss the results.

[16] The results data from the 2023 parliamentary election in Finland were retrieved from the official website of the Ministry of Justice: (2023). We additionally used electoral districts (constituencies) population data provided by Statistics Finland's free-of-charge statistical databases (2024). We used the data from December 31, 2022, and assumed the population was the same for the election held in April 2023.

Table 6 Gallagher Index (%) computed for the global political distribution, the average of the local political distributions, and the 3-dimensional seat distribution, for the 2023 Parliamentary Election in Finland

	E23	CCM	TPM	TPM3	TPP	TPP3
Global GI	3.3	3.3	0.9	2.3	0.7	2.2
Local GI avg.	5.7	5.7	14.7	14.8	6.8	7.5
3-dim GI	2.8	2.6	2.0	2.1	2.0	2.1

Table 7 Average votes obtained by the elected candidates under each method, and the average percentage of votes obtained by elected candidates with respect to votes cast in their districts, for the 2023 Parliamentary Election in Finland

	E23	CCM	TPM	TPM3	TPP	TPP3	Greedy
Avg. votes	7508	7388	7206	7244	7223	7297	7603
Avg. district %	49.3	48.5	41.7	41.9	47.6	47.9	49.9

The indicators regarding proportionality are shown in Table 6. The magnitudes for the Gallagher indices differ, but we observe the same trends as in the Chilean election. Local methods, namely E23 and CCM, have higher values relative to the other (global) methods in the global and 3-dimensional Gallagher Index, where the TPP obtains the best results, followed closely by TPM. The incorporation of a 3% threshold naturally worsens the results, as we are systematically excluding a set of participants from the apportionment. We can also observe, like in the Chilean case, how the local methods obtain better results on average at the district level. Despite the district-level indices being similar between methods in most districts, as shown in Table 11, some particular districts have relevant differences and explain the disparities observed in Table 6. District 5, with only one seat to allocate, is an example of this: The four lists competing add up to less than 1% of all votes cast at a national level, so TPM and TPM3 allocate a seat to a candidate from another nationally relevant list for the sake of global proportionality, attaining the maximum possible discrepancy. On the other hand, TPP and TPP3 allocate the seat to the top-voted candidate in the district.

In Table 7, we can observe the representativeness of the election in terms of average votes obtained by elected candidates and the average percentage of votes obtained per district. The magnitudes are different when compared to the Chilean case due to the differences in house size, number of candidates, and votes cast, but the trends are similar. Applying global methods to elections that are designed for electing candidates on a district level naturally affects the average votes: Some candidates with fewer votes may get elected in place of more voted candidates to achieve global proportionality. However, the relative loss in the average votes is no larger than 4% compared to E23. An interesting observation is that in the Finnish election, unlike the Chilean one, the TPP method no longer achieves the highest average vote count per candidate. This is due to the fact that all the top-voted candidates in every district already get elected in the original election, so they do not benefit from the majority rule.

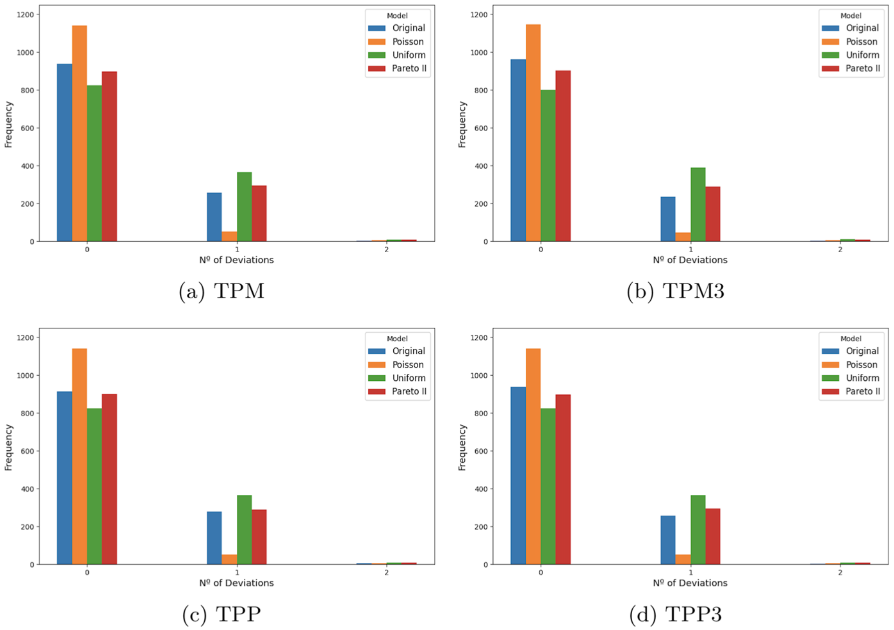


Fig. 5 Distribution of absolute deviation magnitude in gender marginals for the 3-proportional methods, for the 2023 Parliamentary Election in Finland. No deviations in other marginals were ever required. We conduct a simulation of $N = 1200$ random vote instances from 4 different distributions and compute the frequency of the deviations for each method and instance

Regarding voting power in the Finnish election, we again compute the ratio between the number of seats assigned to each district and the number of votes cast and normalize the ratio using the nation-level ratio between seats and votes. These ratios are shown in Table 12. In this case, the ratios range between 0.92 and 1.16, so the relative difference in voting power between districts is at most 1.25, while in the Chilean case it is approximately 5.96. In the case of Finland, the seat assignment by district allows for a more balanced representation of voters' preferences between different districts.

Lastly, we conduct the same simulations as in Sect. 5.4, regarding the deviations required in the 3-proportional methods, for the Finnish election. The results are shown in Fig. 5. We observe that the distributions follow the same pattern as in Fig. 4: In most simulations, no deviations are required for the apportionment, and we never require deviations across marginals other than gender. Furthermore, we do not even require more than two additional seats in the gender marginal. Similarly to the Chilean case, the model based on the Poisson distribution requires the least number of deviations, while the uniform instances require the most.

6 Discussion

In this work, we have studied apportionment methods based on multidimensional proportionality from an applied perspective, establishing practically relevant theoretical results and proposing concrete methods capable of combining this notion with other desirable features of apportionment systems.

As our theoretical contribution, we have established two results that support the usage of multidimensional proportional apportionment methods as proposed by Cembrano et al. (2022). We first showed that, in the fractional case, this notion is the only one satisfying three widely studied axioms, thereby extending a result for two dimensions (Balinski and Demange 1989b) and justifying the corresponding notion of integral proportionality. We have then proved that the addition of bounds on the entries of the apportionment tensor does not affect the existence of approximately proportional apportionments, which is particularly relevant for adding plurality constraints on top of the basic method.

Regarding experimental results, we have used the Chilean Constitutional Convention election as a testing ground to analyze the outcomes of four methods based on multidimensional proportionality in terms of political representation, representativeness, and how well they accomplish the ideal of *one person, one vote*. These methods consist of the basic method proposed by Cembrano et al. (2022) and three methods incorporating one or two additional features: a threshold on the number of votes that a list requires to receive any seats and the election of the most-voted candidate of each district. Furthermore, we have replicated the simulations in the context of the 2023 Parliamentary Election in Finland to support our findings. The methods perform very similarly in both the Chilean and Finnish elections: The patterns in terms of proportionality and representativeness of the methods we considered largely coincide, and the deviations required to obtain an apportionment are consistently bounded by smaller constants than those guaranteed by the theoretical results.

Regarding proportionality in political representation, our results highlight a natural trade-off between global and local representation: Local methods (such as the one used in the election) generate global distortion in list allocation, while global methods based on multidimensional proportionality generate district distortion. In this trade-off, the incorporation of plurality constraints presents a middle ground, as highly-voted candidates who run as part of local projects are not excluded due to obtaining too few votes on a national scale. When considering the average vote of elected candidates, this method exhibits the best performance, while the threshold slightly lowers the average votes. Regarding voting power among districts, we observe that the ratio between votes and seats varies considerably across districts. This effect is attenuated in methods based on multidimensional proportionality, as the allocation of seats to lists is mainly given by the national amount of votes a list gets. Overall, both the threshold and the plurality constraints achieve their respective goals of favoring larger political coalitions and incorporating some degree of local representation. The cost of the former, in terms of proportionality and representativeness, seems to be larger than that of the latter in our simulations, though the exact way in which they are applied may of course have different impacts.

The presented methods can be valuable and feasible proposals for the election of representative bodies. Proportional or mixed-member proportional representation systems are two families of electoral systems widely used worldwide, and some countries have also started to use biproportional apportionment methods (Maier et al. 2010; Pukelsheim 2017). Our theoretical and applied results lay the groundwork for incorporating features such as representation threshold—to favor the conformation of broader political projects—or the election of the most-voted candidate in every district—to enhance local representation—into methods based on multidimensional proportionality. In particular, a method incorporating both features, such as TPP3, may constitute an appealing alternative to balance local and global proportionality, representativeness, and voting power.

Although multidimensional proportional methods may appear harder to understand for citizens at first glance, all methods incorporating additional constraints such as gender balance suffer from this problem, and the concepts of proportionality, gender balance, and plurality directly imposed in the proposed methods are certainly intuitive in contrast with the somehow subtle corrections made in the actual method used in the election of the Chilean Constitutional Convention. It is also important to mention that even though the ease of understanding constitutes a relevant element in terms of the legitimacy of the electoral process, a proper description and an appropriate performance of the mechanism do as well. Moreover, 3-proportional methods are easily extended to more dimensions to include, for instance, ethnicity as a fourth dimension and thus allow ethnic seats to be proportionally assigned across political, geographical, and gender distribution. We remark that methods based on multidimensional proportionality are computationally efficient, both theoretically and in practice. Furthermore, the observed deviations from the prescribed marginals required to implement these methods are, in most cases, zero or very small.

All these facts consolidate the possibility of thinking deeper about electoral methods incorporating several dimensions and constraints. This future work certainly requires an interdisciplinary approach in order to succeed in designing mechanisms able to better represent the complexity and diversity of modern societies. For example, this work does not account for the effects of the new rules on the voters' and parties' strategic behavior. Another direction concerns the relationship of the electoral rules with the legitimacy of the elected representative bodies. From a mathematical and computational perspective, this work shows that optimization and algorithmic tools are valuable for designing and testing new electoral methods for increasingly complex societies, and motivates the development of further approaches.

Appendix A Tables

See Tables 8, 9, 10, 11 and 12.

Table 8 Percentage of votes obtained by list, and proportion of seats obtained by list under each method for the Chilean election

	Votes	CCM	BPM	TPM	TPM3	TPP	TPP3
XP	22.8	26.8	26.8	23.9	25.4	23.2	24.6
YQ	20.7	20.3	20.3	21.7	23.2	21.0	22.5
LP	18.3	19.6	19.6	19.6	20.3	18.8	19.6
YB	16.0	18.1	18.1	16.7	18.1	16.7	17.4
INN	8.7	8.0	8.0	8.7	9.4	8.7	9.4
XA	3.8	0.0	0.0	3.6	3.6	3.6	3.6
ZR	1.0	0.0	0.0	0.7	0.0	0.7	0.0
ZB	0.8	0.0	0.0	0.7	0.0	0.7	0.0
YK	0.8	0.7	0.7	0.7	0.0	0.7	0.0
YU	0.8	0.0	0.0	0.7	0.0	0.7	0.0
ZL	0.8	0.0	0.0	0.7	0.0	0.7	0.0
T	0.7	0.7	0.7	0.7	0.0	0.7	0.0
YX	0.7	0.0	0.0	0.7	0.0	0.7	0.0
ZW	0.7	0.7	0.7	0.7	0.0	0.0	0.0
IND9	0.7	0.7	0.7	0.0	0.0	0.7	0.7
XI	0.7	0.7	0.7	0.0	0.0	0.0	0.0
ZZ	0.6	0.7	0.7	0.0	0.0	0.7	0.7
WB	0.5	0.7	0.7	0.0	0.0	0.0	0.0
P	0.4	0.7	0.7	0.0	0.0	0.0	0.0
A	0.2	0.7	0.7	0.0	0.0	0.0	0.0
XM	0.2	0.7	0.7	0.0	0.0	0.7	0.7
IND1	0.1	0.0	0.0	0.0	0.0	0.7	0.7

Table 9 One-dimensional Gallagher index computed for each district, for the Chilean election

	CCM	TPM	TPM3	TPP	TPP3
1	0.22	0.24	0.24	0.28	0.28
2	0.15	0.29	0.29	0.29	0.29
3	0.23	0.24	0.32	0.20	0.13
4	0.13	0.19	0.19	0.19	0.19
5	0.06	0.14	0.14	0.14	0.14
6	0.08	0.11	0.14	0.11	0.14
7	0.08	0.08	0.08	0.08	0.08
8	0.11	0.12	0.20	0.12	0.20
9	0.12	0.20	0.20	0.17	0.17
10	0.09	0.10	0.17	0.10	0.17
11	0.14	0.14	0.27	0.14	0.27
12	0.11	0.11	0.16	0.11	0.16
13	0.32	0.19	0.19	0.19	0.19
14	0.08	0.18	0.18	0.18	0.18
15	0.10	0.22	0.21	0.26	0.21
16	0.08	0.08	0.08	0.08	0.08
17	0.08	0.18	0.18	0.18	0.18
18	0.17	0.26	0.17	0.17	0.17
19	0.15	0.18	0.18	0.18	0.18
20	0.09	0.17	0.20	0.20	0.20
21	0.11	0.20	0.11	0.20	0.11
22	0.28	0.28	0.18	0.28	0.18
23	0.09	0.11	0.09	0.11	0.15
24	0.20	0.21	0.21	0.21	0.21
25	0.28	0.17	0.17	0.17	0.17
26	0.13	0.23	0.23	0.23	0.23
27	0.16	0.29	0.29	0.29	0.29
28	0.24	0.25	0.25	0.24	0.24

Table 10 Percentage of votes obtained by list, and proportion of seats obtained by list under each method for the Finnish election

	Votes	E23	CCM	TPM	TPM3	TPP	TPP3
FINNS	20.5	23.0	23.0	21.0	21.5	21.0	21.5
SDP	20.4	21.5	21.5	21.0	21.5	20.5	21.5
NCP	17.0	19.0	19.0	17.5	18.0	17.5	17.5
CENT	11.3	11.5	11.5	11.5	11.5	11.5	11.5
LEFT	7.2	5.5	5.5	7.0	7.5	7.0	7.5
GREEN	7.2	6.5	6.5	7.0	7.5	7.0	7.5
SPP	4.4	4.5	4.5	4.0	4.5	4.0	4.5
KOK+RKP	3.9	4.5	4.5	4.0	4.0	4.0	4.0
CD	3.8	2.5	2.5	3.5	4.0	3.5	4.0
MOVE	2.5	0.5	0.5	2.5	0.0	2.5	0.0
SKE+KRIP+VL	0.6	0.0	0.0	0.5	0.0	0.5	0.0
LIBE	0.5	0.0	0.0	0.5	0.0	0.5	0.0
KOK+KD	0.4	0.5	0.5	0.0	0.0	0.0	0.0
FÅ	0.4	0.5	0.5	0.0	0.0	0.5	0.5

Table 11 One-dimensional Gallagher index computed for each district, for the Finnish election

	E23	CCM	TPM	TPM3	TPP	TPP3
1	0.03	0.03	0.06	0.06	0.05	0.06
2	0.03	0.03	0.05	0.05	0.05	0.05
3	0.06	0.06	0.08	0.08	0.08	0.08
4	0.11	0.11	0.09	0.09	0.09	0.11
5	0.00	0.00	1.00	1.00	0.00	0.00
6	0.06	0.06	0.06	0.06	0.06	0.06
7	0.06	0.06	0.05	0.05	0.04	0.05
8	0.06	0.06	0.07	0.09	0.06	0.09
9	0.04	0.04	0.05	0.05	0.05	0.08
10	0.06	0.06	0.06	0.06	0.06	0.07
11	0.08	0.08	0.08	0.08	0.08	0.08
12	0.03	0.03	0.03	0.03	0.03	0.03
13	0.11	0.11	0.22	0.22	0.22	0.22

Table 12 Voting power per district, measured as the ratio between district seats and votes cast in the district, normalized by the nation-level ratio

	Votes	Seats	Ratio
1	388,501	23	0.92
2	565,306	37	1.01
3	278,772	17	0.94
4	119,611	8	1.04
5	13,383	1	1.16
6	207,266	14	1.05
7	309,941	20	1.00
8	228,628	15	1.02
9	224,357	15	1.03
10	246,966	16	1.00
11	153,254	10	1.01
12	261,715	18	1.06
13	97,904	6	0.95

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Data availability Data for the 2021 Chilean Constitutional Convention election is available on the website of Servicio Electoral de Chile (https://www.servel.cl/centro-de-datos/resultados-electorales-historicos-gw3/). Data for the 2023 parliamentary election in Finland is available on the website of the Ministry of Justice (https://tulospalvelu.vaalit.fi/EKV-2023/en/ladattavat_tiedostot.html).

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